Chaotic Hamiltonian systems: Survival probability

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We consider the dynamical system described by the area-preserving standard mapping. It is known for this system that $P(t)$, the normalized number of recurrences staying in some given domain of the phase space at time t (so-called "survival probability") has the power-law asymptotics, $P(t) \sim t^{-\nu}$. We present new semiphenomenological arguments which enable us to map the dynamical system near the chaos border onto the effective "ultrametric diffusion" on the boundary of a treelike space with hierarchically organized transition rates. In the framework of our approach we have estimated the exponent ν as $\nu = \ln 2/\ln(1+r_g) \approx 1.44$, where $r_g = (\sqrt{5}-1)/2$ is the critical rotation number.

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I. INTRODUCTION

In this paper we propose a new estimate for the "Poincaré recurrences" (or "survival probability"), $P(t)$, for the standard mapping in the vicinity of the chaos border [1,2]. Our consideration explicitly exploits the fact that the phase space of the standard mapping near the border separating chaotic and integrable behaviors has self-similar scale-invariant structure consisting of hierarchical set of metastable islands of integrability ("cantori") embedded in the "chaotic sea" [3]. To be specific, we consider the area-preserving standard mapping

$$y_{t+1} = y_t - K/(2\pi)\sin 2\pi x_t,$$

$$x_{t+1} = x_t + y_{t+1} \quad \mod 1. \qquad (1)$$

For $K \in [0, K_g[$ the phase space of the system has disjoined islands of integrability which is destroyed as $K \to K_g$ from below, where $K_g \approx 0.971\,635\,406\,31$ [3]. Above the critical value K_g the behavior of the system is less universal: many invariant Kolmogorov-Arnold-Moser (KAM) tori disappear, but still some islands of metastability survive around the biggest resonances. However, we should emphasize that the reorganization of the phase space above the value K_g has no consequence for our consideration since we do not touch the region $K > K_g$ and are interested in the survival probability only when K approaches K_g from below.

Our consideration of the survival probability is semiphenomenological, that is we rely only on the measurable "macroscopic" characteristics acquired in course of the iteration of the map [Eq. (1)]. To be precise, we rely on the following well–established and confirmed facts: (i) the number of principal resonances follows the Fibonacci sequence when $K \nearrow K_g$ [4,5]; (ii) the generic behavior of phase trajectories is as follows: the phase trajectory stays in the vicinity of some resonance (low-flux cantori) and then rapidly crosses the chaotic sea until another metastable low-flux cantori is reached [4,6]; (iii) the phase space of the standard mapping is self–similar being usually represented by a binary (i.e., 3-branching) Cayley tree [4,7,8]; (iv) the survival probability has power-law asymptotic behavior, $P(t) \sim t^{-\nu}$ (for the first time this has been shown in [1,2]).

Remind that survival probability is the normalized number of recurrences [Eq. (1)] which stay in some given domain of the phase space at time t. Different research groups present different arguments for estimates of ν, typically $1 < \nu \lesssim 3$. The most intriguing contradiction concerns the discrepancy in reported values of ν. The numerical simulations [2,9] demonstrate $\nu \approx 1.4-1.5$, while almost all known analytic constructions give essentially larger exponents: $\nu \approx 1.96$ in [7] and $\nu \approx 3.05$ in [8]. The scaling analysis [5] valid just near the chaos boundary gives $\nu = 3$. The special attention should be paid to the recent works [10–12]. In [10] the authors demonstrate that by an appropriate randomization of the "Markov tree model" proposed in [7,8] one can arrive at the value $\nu \approx 1.57$. The works [11,12] claim $\nu = 3$ for sticking of trajectories near K_g in agreement with [5] and $\nu = 3/2$ for trapping of chaotic trajectories in the vicinity of cantori for $K \approx 2\ell/\pi$ (where ℓ is a nonzero integer). The similar exponent, $\nu = 3/2$, was also obtained for a standard map in the work [13]. There is a point of view that the value $\nu \approx 1.4-1.5$ corresponds to an intermediate behavior of the system which is not yet reached the stationary regime—see the corresponding discussion in [5,14–16].

The main aim of the present letter is to present some simple arguments in favor of the statement that the value $\nu \approx 1.4-1.5$ could be the actual decay exponent of the survival probability $P(t)$ for $t \to \infty$ without additional randomization of local transitional probabilities. Let us emphasize once more that in our approach we get rid of microscopic consideration of the detailed structure of quasiperiodic orbits and corresponding fluxes, but characterize the phase space of our system for a given coupling constant, K in the vicinity of the critical value K_g, just by the hierarchy of resonances.

According to [3,17,18], the structure of the critical KAM curve is determined by arithmetic properties of the rotation number, r, in the continued fraction representation:

$$r = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{m_3 + \dots}}} = [m_1 m_2 m_3 \dots]. \qquad (2)$$

The ns best approximant to r is $r_n = [m_1 m_2 \dots m_n] = p_n / q_n$ with $m_1 = m_2 = \dots = m_n = 1$, where q_n is the Fibonacci number. Recall that the Fibonacci numbers satisfy the recursion relation $q_{n+1} = q_n + q_{n-1}$; $q_0 = 0, q_1 = 1$.

For $K \nearrow K_g$ the periodic trajectories with rotation numbers r_n determine the structure of the phase space, converging to the critical boundary curve with r_g (see [3]). In the limit $n \to \infty$ (i.e., for $q_n \to \infty$) the phase space becomes self-similar with the scaling factor $s_n = q_n / q_{n-1} \to 1 + r_g \approx 1.618$, where $r_g = [111\dots] = (\sqrt{5}-1)/2$ is the "golden mean." The convergents r_n (n is fixed) characterize the positions of unstable fixed points of resonances for a given value of a coupling constant, K.

According to [5], the average local exit time, τ_n, from a given scale n can be estimated as $\tau_n \sim |r_g - r'_n|^2 / D_n$, where r'_n is the ns convergent of the critical rotation number, r_g, and D_n is the local diffusion coefficient. Since $|r_g - r'_n| \sim q_n^{-2}$ and $D_n \sim q_n^{-5}$, one gets the estimate for the local exit time,

$$\tau_n \sim q_n. \qquad (3)$$

When $n \to \infty$ (i.e., when $K \to K_g$) one finds for the exit time, τ, from the scale n the following asymptotic behavior:

$$\tau_n \simeq (1 + r_g)^n = (1.618)^n. \qquad (4)$$

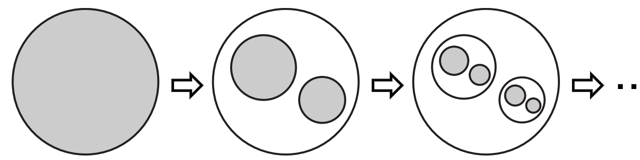
To summarize, the following two facts [5] constitute the basis of our semi–phenomenological consideration:

(i) When approaching the chaos boundary from below, the average local exit time, τ_n, from the metastable island of the hierarchy scale n is proportional to the number of periodic orbits, q_n [see Eq. (3)]. This fact reflects the "local" structure of our phase space;

(ii) When $n \to \infty$, the local exit time, τ_n, grows exponentially with the hierarchy scale n [see Eq. (4)]. This fact reflects the "global" structure of our phase space and allows to construct the estimate for the survival probability.

II. DYNAMICS IN THE HIERARCHICAL TREELIKE PHASE SPACE: DIFFUSION ON THE BOUNDARY VS DIFFUSION IN THE BULK

The hierarchical construction of self-similar sets implies that new smaller domains being properly magnified (rescaled) with the limiting scaling factor s_g, coincide (in the statistical sense) with the former ("parent") domains. This construction refers implicitly to the treelike geometry. The treelike hierarchical geometry of the phase space of the standard mapping has been proposed in the seminal analytic works [7,8]. The authors have supposed that the states of the system are the regions bounded by the low-flux cantori. Each state consists of an infinite hierarchy of low-flux cantori of smaller scale. Between any two adjacent cantori there are possible many other subhierarchies (or "island chains" in the terminology of [7]). However only one such "island chain" in each hierarchical level is considered. Thus, the topology of the full phase space can be represented as a following diagrammatic hierarchy, where the hierarchy depth (level) is labeled by the same index n, appeared already in Eq. (2) as the cutoff in the continuous fraction expansion:



Knowing the local transition rates between neighboring states from microscopic computations, the authors in [7,8] have considered the random walk on the 3-branching tree and have derived the corresponding infinite hierarchy of recursive relations for the survival probability. The plausible conjectures about the closure of this hierarchy and subsequent analysis of the dominant contribution to $P(t)$ allowed to extract the decay exponent $\nu \approx 1.96$ in [7]. However this value is still rather far from numerically obtained exponent $\nu \approx 1.4-1.5$. Below we formulate an alternative point of view on dynamics on hierarchical landscapes enabling to find the value of ν much closer to known numerical value of the exponent ν.

The hierarchical description sketched above is fully consistent with the consideration of a "diffusion in a mountain landscape" [19] appeared in a generic description of "diffusion in hierarchies" regardless the subject of the chaotic Hamiltonian dynamics. For heuristic consideration, let two sequences of real numbers, A_n and V_n, be correspondingly the sizes of landscape valleys (basins) and the heights of passages (energy barriers) between these valleys with respect to some reference (energy) level. The basin sizes and the barrier heights can be introduced iteratively. Suppose that some dynamical system is located in a basin A_0 at the initial time moment. During the time t_1 the system overpasses the lowest available height V_1 and the probability to find the system in some state distributed initially on A_0, relaxes to the larger basin $A_1 \supset A_0$. Inductively, the probability to find the system located in the basin A_n at time t_n, relaxes at time t_{n+1} to a larger domain $A_{n+1} \supset A_n$ by surmounting the lowest available height V_n. Since by construction the n's basin hierarchically includes all basins $A_{n-1} \supset A_{n-2} \supset \dots \supset A_0$, there is no difference between the waiting time in the n's basin and the total time from the beginning of the dynamical process. Thus, the survival probability under diffusion on hierarchies depends essentially only on scaling of basin sizes, A_n, and barrier heights, V_n, and exactly this fact allows one to use the treelike spaces with alternative (Archimedean or non-Archimedean) metrics (see, for example [20,21]).

Now we are in position to describe the main idea of the present work. In our description the state of the system for some value of K near the chaos border is uniquely characterized by the number of quasiperiodic orbits, q_n at the hierarchy level n. Thus, the states are parameterized by the Fibonacci numbers, q_n.

We consider the dynamics of the system as the transitions between different states for *given* value of n. This is the key difference with the former description of [7,8] schematically outlined above. Namely, in the former approach the authors have considered the local random walk in the *bulk* of the 3-branching Cayley tree with the transitions between neighboring states belonging to two neighboring levels of hierarchy. Such a construction suggests an Archimedean metric on the treelike space. To the contrary, we propose to consider an effective Markov dynamics on the *boundary* of the 3-branching Cayley tree truncated at the hierarchy level n ($n \gg 1$). We allow for *long distance* jumps which appear with the probability prescribed by the limiting scaling factor s_g *only between the states of the same hierarchy level n*. This construction implicitly suggests non-Archimedean (ultrametric) space of states.

To be precise, our Markov process takes place in an effective "energy landscape" constructed in the following way. We consider the phase space up to the scale n meaning that we regard the low-flux cantori (metastable islands) of scale n as local (possibly degenerated) minima of an energy landscape. Introduce now the self-similar scale-invariant structure of the *basins* of local minima hierarchically embedded into each others. Namely, each larger basin of minima consists of smaller basins, each of these consists of even smaller ones etc. Since the hierarchy level n is chosen arbitrary, each local minimum (i.e., low-flux cantorus of scale n) could be (and should be) understood as a basin as well, containing again the scale-invariant hierarchy ("subtree") of basins of smaller scale. Such a hierarchy does not conserve the structure of phase space as the islands of stability in the chaotic sea, but preserves the treelike factorization of islands of smaller scale out of the islands of larger scale as $K \nearrow K_g$. Note that in the common description of treelike factorization of low-flux cantori, the value of n is counted from the root of the tree, meaning that larger and larger values of n correspond to metastable islands of smaller and smaller scales.

To specify our long-distance-jump Markov process in terms of the transition rates between the states (local minima), we construct the hierarchy of barriers between the basins of minima. Namely, larger basins are separated by higher barriers, while embedded smaller basins are separated by lower barriers. The Cayley tree can be regarded as a hierarchical "skeleton" of energy landscape (but not as a space of states as in [7,8]): the bulk vertices of the tree parameterize the hierarchy of basins of local minima hierarchically embedded into each others; the same vertices parameterize the barriers separating the basins.

The crucial requirement to our Markov process is as follows: the transition probability (per time unit) between any two local minima is determined by the maximal barrier separating these minima. This requirement is equivalent to the strong triangle inequality. Thus, basin sizes and barrier heights can be expressed in terms of ultrametric distances between local minima.

In the conventional description of ultrametric spaces accepted in applications of p-adic mathematical analysis [22] to dynamics on energy landscapes, the ultrametric distances between local minima are labeled in such a way that smaller transition rates (i.e., higher barriers) correspond to larger ultrametric distances. Therefore, while specifying ultrametric distances similar to the p-adic norm, $p^\gamma, \gamma=1,2,\ldots,n$ (p is the prime number), we define the index γ which numerates

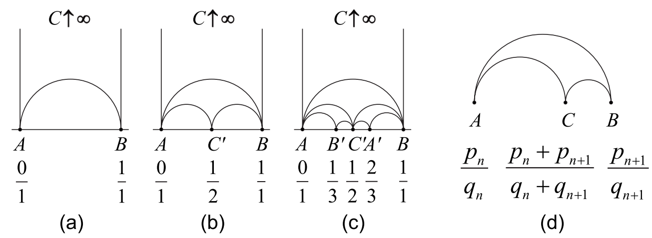


FIG. 1. Inversions of zero-angled triangles: (a)–(c)—three subsequent stages of tessellation of the half-strip $\mathrm{Im}\, z>0; 0\le \mathrm{Re}\, z \le 1$ by the images of the triangle ABC; (d)—composition rule for the coordinates of the vertices.

the hierarchy levels of Cayley tree in the direction opposite to the one of index n. In this construction $\gamma=1,2,\ldots,n$ is counted from the *boundary* of the tree up to the tree origin, namely, there is a hierarchy of ultrametric distances between the states of scale n possessing the values $p^1, p^2, \ldots, p^n$.

The hierarchical landscape with growing barriers corresponds to the fact that the average local exit time τ_n from the states of scale n grows with n: larger the tree (i.e., deeper the hierarchy), longer the exit time averaged over all states on the tree boundary. This condition does not yet defines completely our energy landscape. We should satisfy another important requirement, viz, the average local exit time τ_n from the states of scale n grows as

$$\tau_n \sim (q_n/q_{n-1})^n \simeq s_g^n \tag{5}$$

for $n \to \infty$, where q_n is the n's Fibonacci number and $s_g \approx 1.618$. This means that the energy landscape, being expressed in terms of ultrametric distances between the states of scale n, meets a specific behavior of the average local exit time, τ_n, with n, and the survival probability as well, through the relation to the Fibonacci numbers. This is another key ingredient of our approach.

III. TREELIKE GEOMETRY AND THE FIBONACCI NUMBERS

The procedure of construction of such a hierarchical landscape is described below. Anyway, to be specific, in what follows we shall always keep in mind the 3-branching Cayley tree, $\mathcal{C}$, as an example of an ultrametric space.

The Fibonacci numbers have natural relation to the ultrametric geometry since they are connected to some discrete symmetries of the hyperbolic space. In order to substantiate our construction, it seems to be instructive to demonstrate briefly how the Fibonacci numbers appear in ultrametric geometry and how they are connected to a 3-branching Cayley tree. Take the upper complex half-plane $z=x+iy$ ($y>0$) and consider the zero-angled curvilinear triangle ABC bounded by two vertical lines AC, BC and a semicircle AB leaned on the real axis as shown in the figure Fig. 1(a). Tessellate now the upper half-plane strip $\mathrm{Im}\, z>0; 0\le \mathrm{Re}\, z\le 1$ by the images of the triangle ABC. Two subsequent steps are shown in Figs. 1(b) and 1(c). These images are obtained by sequential inversions (fractional-linear transformations) of the initial triangle ABC. The images of all vertices of the triangle ABC

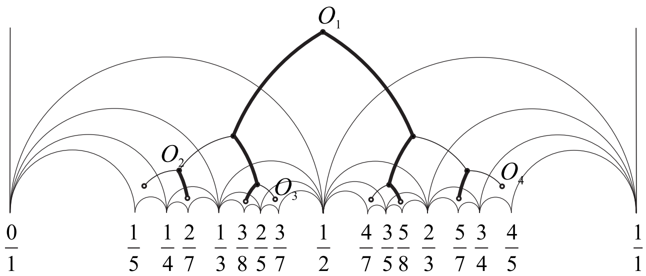


FIG. 2. Few subsequent inversions of the zero–angled triangle in the strip Im $z>0;0\leq\text{Re } z\leq 1$ is shown). Inversions, corresponding to Fibonacci sequences are shown in boldface.

lie on the real axis Im $z=0$ and the coordinates of the vertices of neighboring triangles satisfy the following composition rule

$$x_{n+2} = \frac{p_{n+2}}{q_{n+2}} \equiv \frac{p_n}{q_n} \oplus \frac{p_{n+1}}{q_{n+1}} \overset{\text{def}}{=} \frac{p_n + p_{n+1}}{q_n + q_{n+1}} \tag{6}$$

shown in Fig. 1(d).

Define now the 3-branching Cayley tree isometrically embedded in the upper half-plane strip Im $z>0;0\leq\text{Re } z\leq 1$ by connecting the centers of neighboring images of zero-angled triangles by arcs being parts of semicircles leaned against the real axis—see the Fig. 2. Recall that the embedding of a Cayley tree $\mathcal{C}$ into the metric space is called "isometric" if $\mathcal{C}$ covers that space, preserving all angles and distances. For example, the rectangular lattice isometrically covers the Euclidean plane $E\{x,y\}$ with the flat metric $ds_E^2=dx^2+dy^2$. In the same way the Cayley tree $\mathcal{C}$ isometrically covers the surface of the constant negative curvature, $\mathcal{H}$. One of possible representations of $\mathcal{H}$, known as a Poincaré model, is the upper half-plane Im $z>0$ of the complex plane $z=x+iy$ endowed with the metric $ds_{\mathcal{H}}^2=(dx^2+dy^2)/y^2$ of constant negative curvature. The composition rule (6) defines the coordinates of corresponding triangle vertices, $\{x_n\}$. Hence, the rule (3) defines a set of rational numbers parameterizing all bulk vertices of the Cayley tree of n hierarchical levels. The Fibonacci sequences appear as the subsets of $\{x_n\}$ corresponding to alternating left-right (or symmetric right-left) sequences of reflections of triangles (these sequences are marked in Fig. 2 in boldface). For example, two "zigzags" starting from the point O_1 correspond to two "principal" Fibonacci sequences, $f^{+}(O_1)=[1111\ldots]$ and $f^{-}(O_1)=1-[1111\ldots]$. The "secondary" zigzags, $f(O_2)$, $f(O_3)$ and $f(O_4)$ starting from the points O_2, O_3, and O_4 correspondingly, have the continued fraction expansions: $f(O_2)=1-[121111\ldots]$, $f(O_3)=[112111\ldots]$, and $f(O_3)=[131111\ldots]$. Generically, all "zigzags" have the following continued fraction representation,

$$f(\text{zig}) = \begin{cases} [1m_2m_3\ldots m_k1111\ldots] & \text{for } 1/2<x<1 \\ 1-[1m_2m_3\ldots m_k1111\ldots] & \text{for } 0<x<1/2 \end{cases} \tag{7}$$

with k first arbitrary numbers $m_2,m_3,\ldots,m_k(m_1=1)$ are followed by the "Fibonacci tail" of "1."

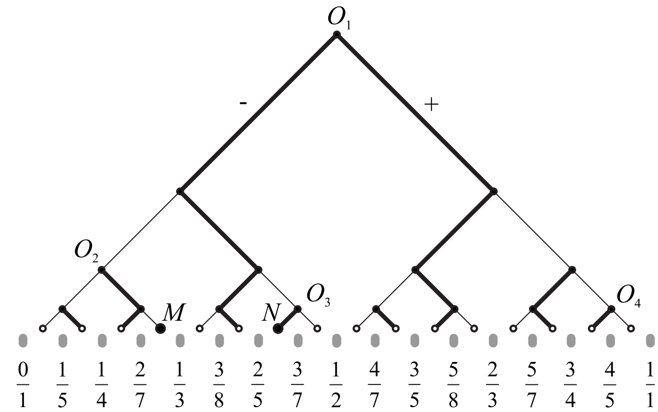


FIG. 3. Topological structure of the graph obtained by the successive applications of fractional-linear transformations in Fig. 2. "Principal" and "secondary" Fibonacci sequences are marked in boldface (see the explanations in the text).

Washing out the metric structure of the set of fractional-linear transformations depicted in Fig. 2 and leaving topological structure of the corresponding "zigzags" on the Cayley tree we arrive at the relation between the Fibonacci numbers and an ultrametric space shown in Fig. 3.

It should be emphasized that the ultrametric graph in Fig. 3 designates schematically the hierarchically organized transition rates (energy barriers) between the states on the tree boundary, while the bottom gray points labeled by rational numbers, p_n/q_n, parameterize the bulk vertices of the ultrametric tree, and hence, they correspond to the transition rates over the barriers. Again, the dynamics occurs only at this boundary, but not in the bulk of the tree.

IV. SURVIVAL PROBABILITY

Let us come back to the computation of the survival probability, $P(t)$. Our main conjecture is as follows. Extend the set of states of the dynamic system and suppose that all fractions p_n/q_n parameterize some quasiperiodic orbits. The transition rates between any two different states M and N are defined by the relative "distance" between these states measured in number of successive reflections (as in Fig. 2) necessary to superpose the points M and N. One might thought about our system in the following terms. Take a 3-branching Cayley tree up to some hierarchical level (generation) n shown in Fig. 3. Parameterize all the bulk vertices of the tree by the sets of rational numbers p_n/q_n following the reflection described above, and consider the "ultrametric diffusion" on the boundary—a Markov process with the transition rates encoded in block-hierarchical kinetic matrix, known as "Parisi transition matrix"—see [21,23]. Thus, our jumplike Markov process is defined by the Parisi kinetic matrix whose matrix elements are encoded by the set of rational numbers p_n/q_n.

The survival probability, $P(t)$, is the probability to find the system in the initial state after t jumps on the boundary of the Cayley tree (with hierarchically organized transition probabilities). According to the works [24] (see also [25] for the transparent geometrical interpretation), the function $P(t)$

consists of additive contributions from all possible directed paths on the p-adic tree (recall that in our case $p=2$),

$$P(t,\Gamma) = (p-1)\sum_{\{\gamma,j\}} p^{-\gamma} e^{\lambda_{\gamma,j}t} + p^{-\Gamma}. \tag{8}$$

The indices γ and j label correspondingly the hierarchical level of the tree $(1 \le \gamma \le \Gamma)$ and the specific point in the hierarchical level γ $(1 \le j \le p^{\Gamma-\gamma})$. Pay attention that now γ is counted from the boundary of the tree toward the root point (see the discussion above). The eigenvalues $\lambda_{\gamma,j}$, are defined via the following construction

$$\lambda_{\gamma,j} = -p^{\gamma} q_{\gamma}^{(j)} - (1-p^{-1}) \underbrace{\sum_{\gamma'=\gamma+1}^{\Gamma} p^{\gamma'} q_{\gamma}^{(j')}}_{\Sigma}, \tag{9}$$

where the sum denoted by Σ runs along the tree from some vertex point labeled by the pair of indices (γ,j) to the root point O_1, and $q_{\gamma}^{(j)}$ is the transition probability corresponding to the state labeled by (γ,j).

The whole variety of directed sequences running from the hierarchical level $\gamma=1$ to the root point O_1 is bounded by two "limiting" trajectories "logarithmic" and "linear." The logarithmic is $\sigma_{\log}=\{1/5, 1/4, 1/3, 1/2\}$, and the linear, being the "principal" Fibonacci one, is $\sigma_{\text{lin}} =\{3/8, 2/5, 1/3, 1/2\}$. The denominators in the logarithmic sequence grow linearly with γ, $q_{\gamma}=\gamma$, while the denominators of the linear sequence grow exponentially, $q_{\gamma} \simeq (1.618)^{\gamma}$. The notations logarithmic and linear come from the fact that $V(\gamma,j) \sim -\ln q_{\gamma}^{(j)}$ can be considered as the effective dimensionless local height of the potential barrier in the point (γ,j). The logarithmic landscape is associated with the logarithmic sequence, for which one has $V(\gamma) \sim \ln \gamma$, while the linear landscape is associated with the linear sequence, for which one has $V(\gamma) \sim \gamma$.

Taking into account that: (i) the eigenvalues (9) of the transition matrix are given by the weighted sums along different directed paths on the Cayley tree, and (ii) the survival probability does not depend on the multiplicity of paths on the tree with the same sequence of transition probabilities, we can directly use the results of [24] for survival probabilities in logarithmic and linear landscapes.

The survival probability for logarithmic and linear landscapes reads (see [24])

$$P(t) \simeq \begin{cases} e^{-t/\ln 2} & \text{for } q_{\gamma} = 2^{-\gamma}\gamma^{-1} \quad \text{(logarithmic)} \\ Ct^{-1/\alpha} & \text{for } q_{\gamma} = 2^{-\gamma}2^{-\alpha\gamma} \quad \text{(linear)} \end{cases}, \tag{10}$$

where $C=\Gamma(\frac{1}{\alpha}+1)(-\Gamma_{p=2}(-\alpha))^{-1/\alpha}$ and $\Gamma_{p=2}(\ldots)$ is the $p=2$-adic Γ function (see [24] for details).

The appearance of the factor $2^{-\gamma}$ in the transition probabilities q_{γ} in Eq. (10) should be clarified. By definition, γ^{-1} (for the logarithmic landscape) and $2^{-\alpha\gamma}$ (for the linear landscape) are the transition probabilities over the barrier of the hierarchy level γ separating two basins. To get the transition probabilities between the *specific points* x (located in one basin) and y (located in the second basin)—just these probabilities enter in the kinetic equation on ultrametric trees—one should divide the transition probability over the barrier by the number of states in the basin, which is in our case 2^{γ}.

The value of α can now be found straightforwardly. Rewrite the transition rate q_{γ} as

$$q_{\gamma} = (1.618)^{-\gamma} = 2^{-\alpha\gamma}. \tag{11}$$

From Eq. (11) one gets

$$\alpha = \frac{\ln 1.618}{\ln 2} \approx 0.6942 \ldots . \tag{12}$$

One sees from Eq. (10) that in the large-time limit $(t\to\infty)$ the decay of the survival probability is exponentially fast on logarithmic landscapes. Since the survival probability consists of additive contributions from all possible directed paths on the 3-adic tree, only the linear landscapes, i.e., the "principal" Fibonacci sequences, give the major contribution to the survival probability in the large-time limit $(t\to\infty)$, leading to the following algebraic decay,

$$P(t) \sim t^{-\nu} = t^{-\ln 2/\ln s_g} \approx t^{-\ln 2/\ln 1.618} = t^{-1.44}, \tag{13}$$

where $s_g=1+r_g=(\sqrt{5}+1)/2 \approx 1.618$. Thus, we have $\nu \approx 1.44$.

V. CONCLUSION

The value $\nu \approx 1.44$, computed in our work, is much closer to numerically obtained critical exponent $\nu \approx 1.4-1.5$ than other values of ν found in former analytic approaches (except the one found in [10–12]). Nevertheless we are far from a naive thought that our phenomenological consideration resolves the problem of analytic computation of the survival probability in the nonlinear dynamical Eq. (1) in a region near the chaos border. We only have reformulated some particular problems of chaotic Hamiltonian dynamics in terms of Markov dynamics on boundaries of ultrametric trees with transition probabilities prescribed by internal dynamics of the system. Recall the two main ingredients of our consideration borrowed from [5]: (i) for $K \nearrow K_g$ the average local exit time, τ_n, from the metastable island of the hierarchy scale n is proportional to the number of periodic orbits, q_n; (ii) for $n\to\infty$ the local exit time, τ_n, grows exponentially with the hierarchy scale n.

Note that our description does not contradict with the value $\nu=3$ found in scaling analysis [5] just at the boundary $K=K_g$. Namely, our consideration "smears" the scaling $\tau_n \simeq s_g^n$ to some region below K_g where the "secondary" Fibonacci sequences starting from the points O_2, O_3, O_4 in the Fig. 3 come into the play.

We conclude this paper by saying that the conjectured approach offers a possibility to rise some interesting questions concerning the internal structure of the standard mapping (1). For example, it would be desirable to check numerically the existence of the "secondary" Fibonacci sequences starting from the points O_2, O_3, O_4 in the Fig. 3. In case of their presence, one would be intriguing to think about the "phyllotaxis" [26,27] in chaotic dynamical systems. By this conjecture we would like to attract the attention of researchers working in nonlinear dynamical and chaotic systems to the language [22] developed for the description of stochastic processes in ultrametric spaces.

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